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Pulse propagation in a spatial and temporal random medium

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Abstract. The method of time-ordered cumulants is used to investigate the behaviour of pulses in a one-dimensional medium in which the phase speed is a random function of space and time. A linear partial differential equation is obtained for the average pulse profile $\langle E(x, t)\rangle$. The dispersion relation between frequency and wavenumber is obtained and used to solve the initial-value problem for the infinite medium.

1. Introduction

It is our purpose in this paper to investigate pulse propagation in a spatially and temporally varying random medium. Part of our interest is in the physical phenomenon itself, while part is in the mathematical technique used. The method of ordered cumulants (Fox 1976) is used to render the problem in an exactly tractable form, which simplifies even further as the result of our assumption that the time correlation is a Dirac delta function.

The principal result of our analysis is equation (3.24), which displays the mean, $\langle E(x, t)\rangle$, as the solution of a hyperbolic, linear partial differential equation. This result is in contrast with other methods (bilocal approximation, two-timing, Feynman diagrams, etc) which necessarily involve approximation procedures (Frisch 1968, Tatarski 1971, Ishimaru 1977).

Extension of the analysis to three dimensions and the development of the Fokker–Planck equation for the covariance $\langle E(x_1, t_1)E(x_2, t_2)\rangle$ are the subject of future publications. Reference is also made to our solution of the corresponding random heat equation (Fox and Barakat 1978).

2. Formulation of problem

Our basic equation is not the wave equation *per se*, but the vector equation

$$\partial \boldsymbol{E}(x, t)/\partial t = c(x, t)\Gamma(\partial \boldsymbol{E}(x, t)/\partial x) \tag{2.1}$$

where $\boldsymbol{E}(x, t)$ is

$$\boldsymbol{E}(x, t) = \begin{vmatrix} E_1(x, t) \\ E_2(x, t) \end{vmatrix} \tag{2.2}$$

and

$$\mathbf{\Gamma} \equiv \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}. \tag{2.3}$$

The speed $c(x, t)$ is taken to be a random function of both x and t. In particular, we take it to be of the form

$$c(x, t) = c_0(1 + \mu(x, t)) \tag{2.4}$$

where $\mu(x, t)$ is a real-valued, zero-mean, stationary gaussian random process with a very small variance, i.e.

$$\langle c(x, t)\rangle = c_0 \qquad \langle \mu(x, t)\rangle = 0 \tag{2.5}$$

$$\langle \mu(x_1, t_1)\mu(x_2, t_2)\rangle = 2R_\mu(x_1 - x_2)\delta(t_1 - t_2). \tag{2.6}$$

The factor two is included for convenience in the analysis and will be factored out in the end; note that $\sigma_\mu^2 \equiv 2R_\mu(0)$. We call attention to the fact that we are allowing μ to be time dependent as well as space dependent.

Equation (2.1), when written out in component form, is

$$\partial E_1/\partial t = c\ \partial E_2/\partial x \qquad \partial E_2/\partial t = c\ \partial E_1/\partial x. \tag{2.7}$$

To obtain a wave equation for E_1 (and E_2) differentiate the first equation with respect to t, the second equation with respect to x, and eliminate E_2:

$$\frac{\partial^2 E_1}{\partial t^2} - c_0^2(1+\mu)^2 \frac{\partial^2 E_1}{\partial x^2} = c_0^2(1+\mu)\frac{\partial \mu}{\partial x}\frac{\partial E_1}{\partial x} + c_0 \frac{\partial \mu}{\partial t}\frac{\partial E_2}{\partial x}. \tag{2.8}$$

The last term on the right-hand side involving E_2 can be eliminated via the first of equations (2.7); the final result is

$$\frac{\partial^2 E}{\partial t^2} - c_0^2(1+\mu)^2 \frac{\partial^2 E}{\partial x^2} = c_0^2(1+\mu)\frac{\partial \mu}{\partial x}\frac{\partial E}{\partial x} + (1+\mu)^{-1} \frac{\partial \mu}{\partial t}\frac{\partial E}{\partial t} \tag{2.9}$$

where we now omit the subscript on E. (This equation is also satisfied by E_2.) The left-hand side is the usual scalar wave equation, but with a random space–time phase velocity. The fact that $c = c(x, t)$ means that the medium is now dispersive. Equation (2.9) is the basic scalar wave equation governing the propagation of $E(x, t)$.

The left-hand side of equation (2.9) set equal to zero is the equation usually studied in the literature with μ depending only on x but not on t. However, second-order differential wave equations are usually derived from two coupled first-order equations such as in equation (2.7). Therefore, equation (2.9) in its entirety is the correct second-order equation when the phase velocity is spatially and temporally modulated. Consequently, our subsequent analysis commences with equation (2.9).

3. Solution by time-ordered cumulants

With the background information, we now return to equation (2.1) and go to an interaction representation

$$\boldsymbol{E}(x, t) = \exp(tc_0\mathbf{\Gamma}\partial_x)\boldsymbol{E}'(x, t) \qquad \partial_x \equiv \partial/\partial x. \tag{3.1}$$

Consequently

$$\begin{aligned}\partial \boldsymbol{E}'(x,t)/\partial t &= \exp(-tc_0\boldsymbol{\Gamma}\partial_x)c_0\mu(x,t)\boldsymbol{\Gamma}\partial_x \exp(+tc_0\boldsymbol{\Gamma}\partial_x)\boldsymbol{E}'(x,t)\\ &= c_0\mu'(x,t)\boldsymbol{\Gamma}(\partial\boldsymbol{E}'(x,t)/\partial x)\end{aligned} \tag{3.2}$$

where

$$\boldsymbol{I}\mu'(x,t) \equiv \exp(-tc_0\boldsymbol{\Gamma}\partial_x)\boldsymbol{I}\mu(x,t)\exp(+tc_0\boldsymbol{\Gamma}\partial x) \tag{3.3}$$

where $\boldsymbol{I}$ is the 2×2 unit matrix. The solution to equation (3.2) is

$$\boldsymbol{E}'(x,t) = \mathcal{T}\exp\left(\int_0^t \boldsymbol{A}(x,s)\,\mathrm{d}s\right)\boldsymbol{E}'(x,0) \tag{3.4}$$

where $\boldsymbol{A}(x,t) \equiv c_0\boldsymbol{\Gamma}\mu'(x,t)\partial_x$ and $\mathcal{T}$ is the time-ordered exponential operator defined by (Fox 1976)

$$\mathcal{T}\exp\left(\int_0^t \boldsymbol{A}(x,s)\,\mathrm{d}s\right) \equiv \boldsymbol{I} + \int_0^t \boldsymbol{A}(x,s)\,\mathrm{d}s + \sum_{n=2}^{\infty}\int_0^t\int_0^{t_1}\ldots\int_0^{t_n}\boldsymbol{A}(x,t_1)\boldsymbol{A}(x,t_2)\ldots\boldsymbol{A}(x,t_n)\,\mathrm{d}t_n\ldots\mathrm{d}t_1. \tag{3.5}$$

We now take the ensemble average of equation (3.4)

$$\begin{aligned}\langle\boldsymbol{E}'(x,t)\rangle &= \left\langle\mathcal{T}\exp\left(\int_0^t \boldsymbol{A}(x,s)\,\mathrm{d}s\right)\right\rangle\boldsymbol{E}'(x,0)\\ &\equiv \mathcal{T}\exp\left(\int_0^t\sum_{n=1}^{\infty}G^{(n)}(s)\,\mathrm{d}s\right)\boldsymbol{E}'(x,0)\end{aligned} \tag{3.6}$$

where $G^{(n)}(s)$ is the nth time-ordered cumulant. Actually this formula *defines* $G^{(n)}(s)$. The explicit relation between $\boldsymbol{A}^{(n)}$ and $G^{(l)}$ is (Fox 1976)

$$\int_0^t \boldsymbol{A}^{(n)}(s)\,\mathrm{d}s = \sum\left[\frac{1}{m_l!}\left(\int_0^t G^{(l)}(s)\,\mathrm{d}s\right)^{m_l}\right]. \tag{3.7}$$

The sum is taken over

$$n = \sum_{l=1}^{\infty} lm_l. \tag{3.8}$$

The integral over $\boldsymbol{A}^{(l)}(s)$ is defined as

$$\int_0^t \boldsymbol{A}^{(l)}(s)\,\mathrm{d}s \equiv \int_0^t\int_0^{t_1}\ldots\int_0^{t_{l-1}}\langle\boldsymbol{A}(t_1)\ldots\boldsymbol{A}(t_l)\rangle\,\mathrm{d}t_l\,\mathrm{d}t_{l-1}\ldots\mathrm{d}t_1. \tag{3.9}$$

Equation (3.7) can be inverted to give $G^{(n)}$ in terms of $\boldsymbol{A}^{(l)}$ (Fox 1976).

It can be shown that

$$\int_0^t G^{(n)}(s)\,\mathrm{d}s = 0 \qquad \text{for } n>2 \tag{3.10}$$

if $\mu(x,t)$ is delta-correlated in time (which is exactly the case we are discussing).

Consequently, we need only concern ourselves with the time integrals of $G^{(1)}(s)$ and $G^{(2)}(s)$. Now

$$\int_0^t G^{(1)}(s)\,\mathrm{d}s = \int_0^t \langle \boldsymbol{A}(t_1)\rangle\,\mathrm{d}t_1 = 0, \tag{3.11}$$

so that equation (3.6) becomes

$$\langle \boldsymbol{E}'(x,t)\rangle = \mathcal{T}\exp\Big(\int_0^t G^{(2)}(s)\,ds\Big)\boldsymbol{E}'(x,0). \tag{3.12}$$

The evaluation of the integral of $G^{(2)}(s)$ can be carried by the sequence of steps

$$\begin{aligned}\int_0^t G^{(2)}(s)\,\mathrm{d}s &= c_0^2\int_0^t\int_0^s \exp(-sc_0\boldsymbol{\Gamma}\partial_x)\langle\boldsymbol{\mu}(x,s)\exp[-(s-s')c_0\boldsymbol{\Gamma}\partial_x]\\ &\quad\times\boldsymbol{\Gamma}\partial_x\boldsymbol{\mu}(x,s')\rangle\exp(+s'c_0\boldsymbol{\Gamma}\partial_x)\boldsymbol{\Gamma}\partial_x\,\mathrm{d}s'\,\mathrm{d}s\\ &= c_0^2\boldsymbol{\Gamma}\int_0^t\int_0^s \exp(-sc_0\boldsymbol{\Gamma}\partial x)I\Big(\frac{\partial}{\partial y}R_\mu(x-y)|_{y=x}\partial_x\Big)\\ &\quad\times\exp(+sc_0\boldsymbol{\Gamma}\partial_x)2\delta(s-s')\,\mathrm{d}s'\,\mathrm{d}s\\ &= c_0^2\boldsymbol{\Gamma}\int_0^t \exp(-sc_0\boldsymbol{\Gamma}\partial_x)I\Big(\frac{\partial}{\partial y}R_\mu(x-y)|_{y=x}\partial_x\Big)\exp(+sc_0\boldsymbol{\Gamma}\partial_x)\,\mathrm{d}s.\end{aligned} \tag{3.13}$$

Upon differentiating equation (3.13) and using (3.12), we have

$$\frac{\partial}{\partial t}\langle\boldsymbol{E}'(x,t)\rangle = c_0^2\boldsymbol{I}\exp(-tc_0\boldsymbol{\Gamma}\partial_x)\Big(\frac{\partial}{\partial y}R_\mu(x-y)\Big|_{y=x}\frac{\partial}{\partial x}\Big)\exp(+tc_0\boldsymbol{\Gamma}\partial_x)\langle\boldsymbol{E}(x,t)\rangle \tag{3.14}$$

which upon transfering back to the original representation via equation (2.11) becomes

$$\frac{\partial}{\partial t}\langle\boldsymbol{E}(x,t)\rangle = c_0\boldsymbol{\Gamma}\frac{\partial}{\partial x}\langle\boldsymbol{E}(x,t)\rangle + c_0^2\boldsymbol{\Gamma}\Big(\frac{\partial}{\partial y}R_\mu(x-y)\Big|_{y=x}\frac{\partial}{\partial x}\Big)\langle\boldsymbol{E}(x,t)\rangle. \tag{3.15}$$

Note that

$$\Big(\frac{\partial}{\partial y}R_\mu(x-y)\Big|_{y=x}\frac{\partial}{\partial x}\Big) = \Big(\frac{\partial}{\partial y}R_\mu(x-y)\Big|_{y=x}\Big)\frac{\partial}{\partial x} + R_\mu(0)\frac{\partial^2}{\partial x^2} \tag{3.16}$$

$$\frac{\partial}{\partial y}R_\mu(x-y)|_{y=x} = 0; \tag{3.17}$$

hence

$$\frac{\partial}{\partial t}\langle\boldsymbol{E}(x,t)\rangle = c_0\boldsymbol{\Gamma}\frac{\partial}{\partial x}\langle\boldsymbol{E}(x,t)\rangle + c_0^2\boldsymbol{\Gamma}R_\mu(0)\frac{\partial^2}{\partial x^2}\langle\boldsymbol{E}(x,t)\rangle. \tag{3.18}$$

To obtain a single partial differential equation for $\langle E_1\rangle$ (or $\langle E_2\rangle$), we first express equation (3.18) in component form:

$$\frac{\partial}{\partial t}\langle E_2\rangle = c_0\frac{\partial}{\partial x}\langle E_1\rangle + c_0^2R_\mu(0)\frac{\partial^2}{\partial x^2}\langle E_2\rangle \tag{3.19}$$

$$\frac{\partial}{\partial t}\langle E_1\rangle = c_0\frac{\partial}{\partial x}\langle E_2\rangle + c_0^2R_\mu(0)\frac{\partial^2}{\partial x^2}\langle E_1\rangle. \tag{3.20}$$

Differentiate equation (3.21) with respect to t, and then substitute equation (3.19) into it:

$$\frac{\partial^2}{\partial t^2}\langle E_1\rangle = c_0\frac{\partial^2}{\partial t\,\partial x}\langle E_2\rangle + c_0^2 R_\mu(0)\frac{\partial^3}{\partial t\,\partial x^2}\langle E_1\rangle$$

$$= c_0^2\frac{\partial^2}{\partial x^2}\langle E_1\rangle + c_0^3 R_\mu(0)\frac{\partial^3}{\partial x^3}\langle E_2\rangle + c_0^2 R_\mu(0)\frac{\partial^3}{\partial t\,\partial x^2}\langle E_1\rangle. \tag{3.21}$$

To eliminate the $\langle E_2\rangle$ term on the right-hand side, differentiate equation (3.20) twice with respect to x and substitute into equation (3.21). The final result is

$$\frac{\partial^2}{\partial t^2}\langle E_1\rangle - c_0^2\frac{\partial^2}{\partial x^2}\langle E_1\rangle - 2c_0^2 R_\mu(0)\frac{\partial^3}{\partial t\,\partial x^2}\langle E_1\rangle - c_0^4 R_\mu^2(0)\frac{\partial^4}{\partial x^4}\langle E_1\rangle = 0. \tag{3.22}$$

We can also show that $\langle E_2\rangle$ satisfies the same equation. At this point we can set $\sigma_\mu^2 = 2R_\mu(0) =$ variance of $\mu(x, t)$ and rewrite equation (3.22) as

$$\frac{\partial^2}{\partial t^2}\langle E\rangle - c_0^2\frac{\partial^2}{\partial x^2}\langle E\rangle - c_0^2\sigma_\mu^2\frac{\partial^3}{\partial t\,\partial x^2}\langle E\rangle + \tfrac{1}{4}c_0^4\sigma_\mu^4\frac{\partial^4}{\partial x^4}\langle E\rangle = 0, \tag{3.23}$$

where we now omit the subscript on E.

Let us rewrite this equation in the form

$$\left(\frac{\partial^2}{\partial t^2} - c_0^2\frac{\partial^2}{\partial x^2}\right)\langle E\rangle - \sigma_\mu^2 c_0^2\frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial t} - \tfrac{1}{4}c_0^2\sigma_\mu^2\frac{\partial^2}{\partial x^2}\right)\langle E\rangle = 0 \tag{3.24}$$

so that the equation consists of the usual deterministic wave equation plus a perturbation term involving a diffusion equation. In accordance with the theory of high-order partial differential equations as developed by Whitham (1974), the highest-order temporal derivatives govern the short-time transient motion while the lowest-order temporal derivatives govern the long-time motion. Thus equation (3.24) can be considered in the following manner. At short times after the pulse is 'launched', the wavefront of $\langle E\rangle$ propagates at speed c_0 because the randomness of the medium has not had sufficient time to react with the pulse. However, as the time is increased, the random medium manifests itself through the lower-order diffusion term and causes the pulse to become dispersive.

4. Initial-value problem

We now consider the initial-value problem for an infinite medium ($-\infty < x < \infty$). In order to solve this problem we must first obtain the dispersion relation between frequency ω and wavenumber k. Assume that

$$\langle E(x, t)\rangle = \exp[\mathrm{i}(kx - \omega t)] \tag{4.1}$$

and substitute into equation (3.23) with the result

$$\omega^2 + \mathrm{i}c_0^2\sigma_\mu^2 k^2\omega - (c_0^2k^2 + \tfrac{1}{4}c_0^4\sigma_\mu^4 k^4) = 0. \tag{4.2}$$

The solutions of this equation are

$$\omega_1 = c_0k - \tfrac{1}{2}\mathrm{i}c_0\sigma_\mu^2k^2 \qquad \omega_2 = -c_0k - \tfrac{1}{2}\mathrm{i}c_0\sigma_\mu^2k^2. \tag{4.3}$$

Note that the ω's are complex-valued functions of σ_μ^2 which become real-valued when $\sigma_\mu^2 = 0$ as required. At this point it is convenient to use the elegant formalism of Eckart (1948) and write

$$\begin{aligned} H_1(k) &= +c_0 k \qquad H_2(k) = -c_0 k \\ D_1(k) &= D_2(k) = D(k) = \tfrac{1}{2} c_0 \sigma_\mu^2 k^2 \end{aligned} \tag{4.4}$$

where the real functions H_1, H_2 are called the Hamiltonian functions of the wave equation and the real functions D_1, D_2 are called its logarithmic decrements.

The Fourier solution of the initial value problem can be shown to be given by (Eckart 1948)

$$\begin{aligned} \langle E(x,t)\rangle = {} & \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\exp[\mathrm{i}(kx - H_1(k)t]\exp(-D(k)t)\,\mathrm{d}k \\ & + \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\exp[\mathrm{i}(kx - H_2(k)t]\exp(-D(k)t)\,\mathrm{d}k. \end{aligned} \tag{4.5}$$

The integrals represent pulses travelling to the right and left of the origin. At time $t = 0$, we have

$$\langle E(x,0)\rangle = \frac{2}{2\pi}\int_{-\infty}^{\infty} F(k)\exp(\mathrm{i}kx)\,\mathrm{d}k \tag{4.6}$$

so that the spatial spectrum of the initial pulse is

$$F(k) = \tfrac{1}{2}\int_{-\infty}^{\infty} \langle E(x,0)\rangle \exp(-\mathrm{i}kx)\,\mathrm{d}k. \tag{4.7}$$

Thus $F(k)$ is determined by $\langle E(x,0)\rangle$.

The evaluation of the integrals in equation (4.5) can be effected by use of the method of stationary phase. However, we will choose $\langle E(x,0)\rangle$ so that the integrations can be carried out in closed form. To this end we let

$$\langle E(x,0)\rangle = \exp(-x^2/a^2) \qquad a > 0; \tag{4.8}$$

then

$$F(k) = (\tfrac{1}{4}\pi a^2)^{1/2}\exp(-\tfrac{1}{4}a^2k^2). \tag{4.9}$$

With this choice of initial disturbance, equation (4.5) becomes

$$\begin{aligned} \langle E(x,t)\rangle = {} & \frac{a}{4\sqrt{\pi}}\int_{-\infty}^{\infty} \exp[\mathrm{i}(x - c_0 t)k]\exp[-\tfrac{1}{2}(c_0 t\sigma_\mu^2 + \tfrac{1}{2}a^2)k^2]\,\mathrm{d}k \\ & + \frac{a}{4\sqrt{\pi}}\int_{-\infty}^{\infty} \exp[\mathrm{i}(x + c_0 t)k]\exp[-\tfrac{1}{2}(c_0 t\sigma_\mu^2 + \tfrac{1}{2}a^2)k^2]\,\mathrm{d}k. \end{aligned} \tag{4.10}$$

These integrals can be evaluated exactly by reference to a Fourier cosine transform table. Upon defining dimensionless variables $\tau \equiv c_0 t/a$, $l \equiv x/a$, $\beta \equiv c_0\sigma_\mu^2/a$, we have

$$\langle E(l,\tau)\rangle = \tfrac{1}{2}(1+\beta\tau)^{-1/2}\exp[-(l-\tau)^2/(1+\beta\tau)] + \tfrac{1}{2}(1+\beta\tau)^{-1/2}\exp[-(l+\tau)^2/(1+\beta\tau)]. \tag{4.11}$$

These are gaussian-shaped pulses travelling to the right and left of the origin with maxima at $l = \pm\tau$; however, the pulses spread and attenuate with time. This is probably best seen by examination of figure 1.

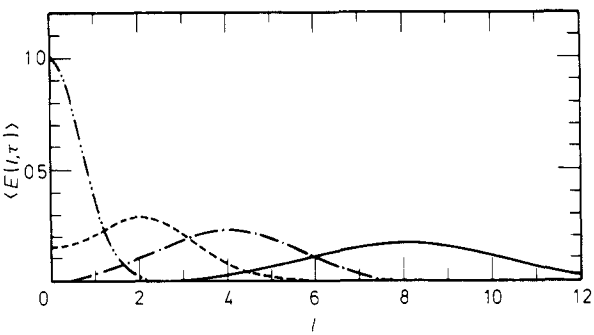

Figure 1. $\langle E(l, \tau)\rangle$ for $\beta = 1$: $- \cdot \cdot -$, $\tau = 0$; $- - -$, $\tau = 2$; $- \cdot -$, $\tau = 4$; ——, $\tau = 8$. Only the right halves of the curves are shown since $\langle E(l, \tau)\rangle = \langle E(-l, \tau)\rangle$.

The area under the pulse is a constant dependent on the initial pulse shape. For our problem

$$\int_{-\infty}^{\infty} \langle E(l, \tau)\rangle \, \mathrm{d}l = (\tfrac{1}{8}\pi)^{1/2}. \tag{4.12}$$

The effect of the small random velocity is to redistribute the energy in the initial pulse but not to dissipate it (at least in this approximation).

5. Comment

The use of a medium in which there is a random process which possesses a delta-function temporal correlation and which is non-delta-function spatially correlated is not new with us. Recently, Lucke (1978) has used randomly stirred fluids with precisely these characteristics in order to study fully developed stationary turbulence. We believe such processes represent an approximation to reality in that the temporal correlations are really non-delta-function correlated, but nevertheless of still very short duration. The linear superposition of many spatically correlated impulses will behave in this manner. For longer-duration temporal correlations, our present results represent the lowest-order non-vanishing cumulant, and corrections are found from the cumulant procedure in a systematic manner. Work is currently progressing along these lines.

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